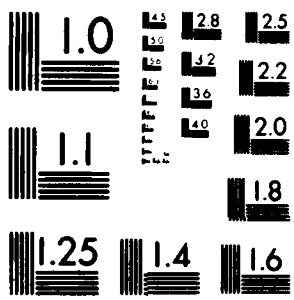
AD-A110 479 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 20/11
ON AN AXISYMMETRIC FREE BOUNDARY PROBLEM.(U)
AUG 81 S ZHOU DAAG29-80-C-0041
UNCLASSIFIED MRC-TSR-2270 NL

END
DATE
FILMED
3 82
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963-A

MRC Technical Summary Report #2270

ON AN AXISYMMETRIC FREE BOUNDARY PROBLEM

Shu-Zi Zhou

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

August 1981

(Received April 8, 1981)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

and

National Science Foundation
Washington, DC 20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

ON AN AXISYMMETRIC FREE BOUNDARY PROBLEM

Shu-Zi Zhou*

Technical Summary Report #2270

August 1981

ABSTRACT

The axisymmetric elastic-plastic torsion of a shaft of general shape subject to the Hencky consistency condition with the von Mises yield function is considered. It is proved that the Haar-Kármán principle is valid in this case, and that the problem is essentially two-dimensional. The problem is reformulated as a variational inequality, and the existence and uniqueness of the solution is studied.

AMS (MOS) Subject Classifications: 35J20; 35J65; 35R35; 73C05; 73E99.

Key Words: Torsion; elastic-plastic; axisymmetric; free boundary problem; variational inequalities; Haar-Kármán principle.

Work Unit Number 1 - Applied Analysis

* Hunan University (Changsha, China) and Mathematics Research Center, University of Wisconsin-Madison.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and the National Science Foundation under Grant No. MCS77-26732.

SIGNIFICANCE AND EXPLANATION

When a shaft of circular cross section is subjected to a torque, a plastic enclave may appear. The boundary between the elastic region and the plastic region is unknown. It is a so-called free boundry problem.

Assuming that Hencky's consistency condition with the von Mises yield function is satisfied, we can prove that the Haar-Kármán principle is valid, which means that the strain energy must be minimized subject to the constraint that the stress should not exceed its permissible limit. We show that the problem is essentially two-dimensional, and give two kinds of variational formulations of the problem: one for the stress field, the other for the stress function. The existence and uniqueness of the solution of the variational inequality for the stress function is proved.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON AN AXISYMMETRIC FREE BOUNDARY PROBLEM

Shu-Zi Zhou*

1. Introduction

The elastic-plastic torsion of shafts is one of the classic free boundary problems. For the case of that the cross-section is constant, it has been studied deeply by using variational inequalities during the last decade (see, for instance, Ting [1971, 1976], Brezis and Sibony [1971], Lanchon [1974], Friedman [1980], Pozzi [1980]). Recently, Cryer [1980] has considered the case in which the shaft has variable cross-section and rotational symmetry. He has proved the existence, uniqueness and regularity of the solution of the variational inequality problem for the stress function under some assumptions. He has assumed that the function it describes the generator of the rotational shaft is monotone. In this paper we consider the case in which the generator may have more general shape; give two kinds of variational formulation of the problem: one for the stress field, the other for the stress function; prove that Haar-Kármán principle is valid and the problem is essentially two-dimensional under the so-called Hencky's conditions; study the existence and uniqueness of the solution.

I am grateful to Professor C. Cryer for many valuable discussions.

* Hunan University (Changsha, China) and the Mathematics Research Center, University of Wisconsin-Madison.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and the National Science Foundation under Grant No. MCS77-26732.

2. Classical problem

Let Figure 1 represent a shaft of circular cross section with equal and opposite pure torques T applied at the ends. Assume that the material is homogeneous, isotropic, and elastic-perfectly plastic, that there are no body forces, and that there are no external tractions on the lateral surface. Our aim is to find the resulting stress distribution.

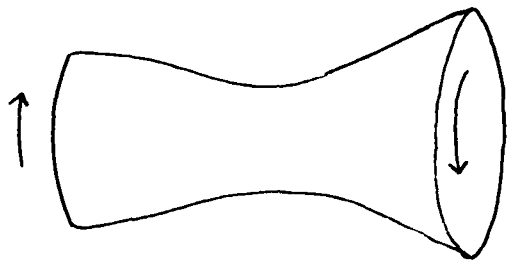


Figure 1

Setting up a cylindrical coordinate system in which the z-axis coincides with the center line of the shaft and the origin lies in a end of the shaft, we assume that the equation for the generator is $r = R(z)$. Let (see Figure 2)

$$\Omega = \{(z,r):\ 0 < z < L,\ 0 < r < R(z)\}$$

$$\Gamma_0 = \{(z,r):\ 0 \leq z \leq L,\ r = 0\}$$

$$\Gamma_1 = \{(z,r):\ 0 \leq z \leq L,\ r = R(z)\}$$

$$\Gamma_{21} = \{(z,r):\ 0 < r < R(0),\ z = 0\}$$

$$\Gamma_{22} = \{(z,r):\ 0 < r < R(L),\ z = L\}$$

$$\Gamma_2 = \Gamma_{21} \cup \Gamma_{22}\ .$$

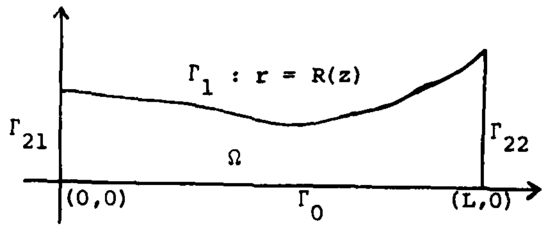


Figure 2

Then we have (Eddy and Shaw [1949]).

<u>Classical Problem</u>: Find stress function $v \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ such that

$$|\nabla v| \le kr^2 \quad \text{in } \Omega \quad (k \text{ is a constant given})$$

$$Av \equiv -\frac{\partial}{\partial z}\left(r^{-3}\frac{\partial v}{\partial z}\right) - \frac{\partial}{\partial r}\left(r^{-3}\frac{\partial v}{\partial r}\right) = 0$$

$$\text{in } \Omega \cap \{(r,z): |\nabla v| < kr^2\}$$

$$v = 0 \text{ on } \Gamma_0, \quad v = T/2\pi \text{ on } \Gamma_1$$

$$\frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_2 \quad .$$

This formulation of the problem is based on the von Mises yield criteria, and on the so-called semi-inverse method given by Saint-Venant (see, for instance, Timoshenko et al. [1951, p. 259, p. 306]) in which one assumes a priori that there are no radial and axial displacement. This fact will be proved in section 4 of this paper.

3. Haar-Kármán principle

The argument we use in these two sections is similar to that in Lanchon [1974] for the case of constant cross section.

From now on we assume that

$$R(z) \text{ is piecewisely, continuously differentiable}$$

$$R'(0) \neq -\infty, \; R'(L) \neq +\infty \tag{3.1}$$

which implies that Ω is strongly Lipschitz domain. Then the three dimensional region Ω^* occupied by the shaft is also strongly Lipschitz domain. Its boundary is

$$\partial\Omega^* = \Gamma_1^* \cup \Gamma_{21}^* \cup \Gamma_{22}^*$$

where Γ_1^* is the lateral surface while Γ_{21}^* and Γ_{22}^* are the end surfaces.

Denote by u_r, u_θ and u_z the components of the displacements in the radial, tangential and axial directions respectively. Let

$$u = [u_r, u_\theta, u_z]^T, \quad \sigma = [\sigma_r, \sigma_\theta, \sigma_z, \sigma_{\theta z}, \sigma_{rz}, \sigma_{r\theta}]^T \quad ,$$
$$\varepsilon = [\varepsilon_r, \varepsilon_\theta, \varepsilon_z, \varepsilon_{\theta z}, \varepsilon_{rz}, \varepsilon_{r\theta}]^T$$

where σ-stress field, ε-strain field. Then we have (see, for instance, Timoshenko et al. [1951, pp. 305-308])

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \quad \varepsilon_z = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}, \quad \varepsilon_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \quad \varepsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

$$\text{in } \Omega^* \qquad (3.2)$$

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} = 0$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \qquad \text{in } \Omega^* \text{ .} \qquad (3.3)$$

We have the boundary condition as follows:

$$\sigma_r n_r + \sigma_{rz} n_z = 0$$
$$\sigma_{r\theta} n_r + \sigma_{\theta z} n_z = 0$$
$$\sigma_{rz} n_r + \sigma_z n_z = 0 \quad \text{on } \Gamma_1^* \text{ (no external tractions)} \qquad (3.4)$$
$$\sigma_{rz} = \sigma_{\theta z} = \sigma_z = 0 \quad \text{on } \Gamma_{21}^* \cup \Gamma_{22}^* \text{ (pure torque)} \qquad (3.5)$$
$$\int_{\Gamma_{21}^*} r\sigma_{\theta z} ds = \int_{\Gamma_{22}^*} r\sigma_{\theta z} ds = T \qquad (3.6)$$

where $(n_r, n_\theta, n_z) = n$ is the outer normal of $\partial\Omega^*$. (Obviously, n is well-defined a.e. on $\partial\Omega^*$ because of (3.1), and $n_\theta = 0$ on $\partial\Omega^*$, $n_r = 0$ on $\Gamma_{21}^* \cup \Gamma_{22}^*$).

Assume that the resulting stress field σ^0 satisfies the Hencky's consistency conditions (Lanchon [1974])

$$\begin{aligned} &F(\sigma^0) \leq 0 \\ &\varepsilon^0 = A\sigma^0 + \lambda \\ &\lambda^T(\sigma-\sigma^0) \leq 0, \qquad \forall\, \sigma \in M_1 \end{aligned} \tag{3.7}$$

where $F(\sigma)$ is the von Mises yield function

$$F(\sigma) = \frac{1}{2}(\sigma^2 + \sigma_{\theta z}^2 + \sigma_{rz}^2 + \sigma_{r\theta}^2) - \frac{1}{6}(\sigma_r + \sigma_\theta + \sigma_z)^2 - k^2 \quad , \tag{3.8}$$

$\lambda = [\lambda_r, \cdots, \lambda_{rz}]^T$, $M_1 = \{\sigma : F(\sigma) \leq 0\}$, and A is the matrix in the Hooke's law (Timoshenko et al. [1951, p. 7, p. 66])

$$A = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & & \\ -\nu & -\nu & 1 & & & \\ & & & 2(1+\nu) & & \\ & & & & (2(1+\nu) & \\ & & & & & 2(1+\nu) \end{bmatrix} \tag{3.9}$$

where E is the modulus of elasticity and ν is the Poisson's ratio with $0 < \nu < 1/2$.

Let

$$M_2 = \{\sigma \in M_1 : \sigma \in [H^1(\Omega^*)]^6,\ \sigma \text{ satisfies } (3.3) - (3.6)\}$$

where the σ in (3.4) - (3.6) is the trace of σ on $\partial\Omega^*$ (see, for instance, Nečas [1967, p. 15]).

<u>Proposition 3.1</u>. If (3.2) and (3.7) are valid for $\sigma^0 \in M_2$, then the Haar-Kármán principle is valid in this case, that is

$$J_1(\sigma^0) = \min_{\sigma \in M_2} J_1(\sigma)$$

where

$$J_1(\sigma) = \frac{1}{2} \int_{\Omega^*} \sigma^T A \sigma dV \quad . \tag{3.10}$$

<u>Proof</u>: $\forall \sigma \in M_2$, we have (noting that A is symmetric)

$$J_1(\sigma) - J_1(\sigma^0) = \frac{1}{2} \int_{\Omega^*} (\sigma^T A \sigma - (\sigma^0)^T A \sigma^0) dV$$

$$= \frac{1}{2} \int_{\Omega^*} (\sigma - \sigma^0)^T A (\sigma - \sigma^0) dV + \int_{\Omega^*} (\sigma^0)^T A (\sigma - \sigma^0) dV \quad .$$

It follows from Gerschgorin's theorem (Varga [1962, p. 16]) that all of the eigenvalues of A are not less than $(1-2\nu)/E > 0$. Hence

$$x^T A x \geq (1-2\nu) x^2/E, \qquad \forall x \in R^6 \tag{3.11}$$

and we have

$$J_1(\sigma) - J_1(\sigma^0) \geq \int_{\Omega^*} (\sigma^0)^T A (\sigma - \sigma^0) dV$$

$$\geq \int_{\Omega^*} (\varepsilon^0)^T (\sigma - \sigma^0) dV \qquad \text{(since (3.7))}$$

$$= \int_{\Omega^*} [\frac{\partial u_r^0}{\partial r} (\sigma_r - \sigma_r^0) + (\frac{u_r^0}{r} + \frac{1}{r} \frac{\partial u_\theta^0}{\partial \theta})(\sigma_\theta - \sigma_\theta^0) + \frac{\partial u_z^0}{\partial z} (\sigma_z - \sigma_z^0)$$

$$+ (\frac{\partial u_\theta^0}{\partial z} + \frac{1}{r} \frac{\partial u_z^0}{\partial \theta})(\sigma_{\theta z} - \sigma_{\theta z}^0) + (\frac{\partial u_r^0}{\partial z} + \frac{\partial u_z^0}{\partial r})(\sigma_{rz} - \sigma_{rz}^0) +$$

$$+ (\frac{1}{r} \frac{\partial u_r^0}{\partial \theta} + \frac{\partial u_\theta^0}{\partial r} - \frac{u_\theta^0}{r})(\sigma_{r\theta} - \sigma_{r\theta}^0)] r dr d\theta dz \qquad \text{(since (3.2))}$$

$$= \int_{\partial\Omega^*} (u_r^0((\sigma_r - \sigma_r^0)n_r + (\sigma_{rz} - \sigma_{rz}^0)n_z) + u_\theta^0((\sigma_{r\theta} - \sigma_{r\theta}^0)n_r + (\sigma_{\theta z} - \sigma_{\theta z}^0)n_z)$$

$$+ u_z^0((\sigma_{rz} - \sigma_{rz}^0)n_r + (\sigma_z - \sigma_z^0)n_z)) dS - \int_{\Omega^*} (u_r^0 (\frac{\partial}{\partial r} (\sigma_r - \sigma_r^0) +$$

$$\frac{1}{r} \frac{\partial}{\partial \theta} (\sigma_{r\theta} - \sigma_{r\theta}^0) + \frac{\partial}{\partial z} (\sigma_{rz} - \sigma_{rz}^0) + \frac{(\sigma_r - \sigma_r^0) - (\sigma_\theta - \sigma_\theta^0)}{r}) + u_\theta^0 (\frac{\partial}{\partial r} (\sigma_{r\theta} - \sigma_{r\theta}^0) +$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}(\sigma_\theta-\sigma_\theta^0) + \frac{\partial}{\partial z}(\sigma_{\theta z}-\sigma_{\theta z}^0) + \frac{2(\sigma_{r\theta}-\sigma_{r\theta}^0)}{r}\Big) + u_z^0\Big(\frac{\partial}{\partial r}(\sigma_{rz}-\sigma_{rz}^0) +$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}(\sigma_{\theta z}-\sigma_{\theta z}^0) + \frac{\partial}{\partial z}(\sigma_z-\sigma_z^0) + \frac{1}{r}(\sigma_{rz}-\sigma_{rz}^0)\Big)\Big)\,dV \qquad \text{(Green's formula)}$$

$$= 0 \quad . \qquad \text{(since (3.3) - (3.5))}$$

Q.E.D.

About more general discussion on the Haar-Kármán principle, see Martin [1975, pp. 733-736].

4. Variational formulation of the problem

Proposition 3.1 suggests the following variational formulation of our problem.

Problem (A) Find $\sigma^0 \in M_2$ such that

$$J_1(\sigma^0) = \min_{\sigma\in M_2} J_1(\sigma)$$

where J_1 is defined by (3.10).

It is easy to show the following assertion.

Proposition 4.1. Problem (A) is equivalent to the variational inequality

$$\begin{aligned} &\sigma^0 \in M_2 \\ &a_1(\sigma^0, \sigma-\sigma^0) \geq 0 \qquad \forall\, \sigma \in M_2 \end{aligned} \tag{4.1}$$

where

$$a_1(\sigma^1,\sigma^2) = \int_{\Omega^*} (\sigma^1)^T A\, \sigma^2\, dV \quad . \tag{4.2}$$

Proposition 4.2. Problem (A) has at most one solution.

Proof: If σ^0 and σ^1 are solutions, then we have by (4.1)

$$a_1(\sigma^0, \sigma^1-\sigma^0) \geq 0$$

$$a_1(\sigma^1, \sigma^0-\sigma^1) \geq 0 \quad ,$$

By virtue of (3.11) we obtain

$$(1-2\nu)\int_{\Omega^*}(\sigma^1-\sigma^0)^2 dV/E \leq a_1(\sigma^1-\sigma^0,\ \sigma^1-\sigma_0)$$

$$= -a_1(\sigma^1,\ \sigma^0-\sigma^1) - a_1(\sigma^0,\ \sigma^1-\sigma_0) \leq 0 \quad .$$

Hence $\sigma^1 = \sigma^0$ a.e. in Ω^*.

Q.E.D.

<u>Theorem 4.3</u>. Assume problem (A) has solution σ^0. Then

$$\text{(a)} \quad \sigma_r^0 = \sigma_\theta^0 = \sigma_z^0 = \sigma_{rz}^0 = 0 \qquad \text{a.e. in } \Omega^*$$

$$\text{(b)} \quad \frac{\partial\sigma^0}{\partial\theta} = 0 \qquad \text{a.e. in } \Omega^*$$

<u>Proof</u>: Let $\sigma^* = [0,\ 0,\ 0,\ \sigma^*_{\theta z},\ 0,\ \sigma^*_{r\theta}]^T$ in $\overline{\Omega^*}$ and

$$\sigma^*_{\theta z} = \frac{1}{2\pi}\int_0^{2\pi}\sigma^0_{\theta z}\,d\theta \qquad \text{in } \overline{\Omega^*}$$

$$\sigma^*_{r\theta} = \frac{1}{2\pi}\int_0^{2\pi}\sigma^0_{r\theta}\,d\theta \qquad \text{in } \overline{\Omega^*} \quad .$$

If we can prove $\sigma^* \in M_2$ and

$$J_1(\sigma^*) \leq J_1(\sigma^0) \quad , \tag{4.3}$$

then the conclusion of the theorem is clear by Proposition 4.2.

At first we prove that $\sigma^* \in [H^1(\Omega^*)]^6$. Recall that Ω is defined by (2.1) and it is the cross-section of Ω^* by plane θ=constant. Define space

$$L_r^2(\Omega) = \{v : v \text{ measurable in } \Omega,\ \|v\|_{L_r^2(\Omega)} < \infty\}$$

with norm

$$\|v\|_{L_r^2(\Omega)} = \int_\Omega v^2 r dr dz \quad ,$$

and define a distribution

$$\langle\sigma^*_{\theta z},\ v\rangle_{\Omega,r} = \int_\Omega \sigma^*_{\theta z}\, v r dr dz, \qquad \forall\ v \in C_0^\infty(\Omega) \quad .$$

$\forall\ \varphi \in C_0^\infty(\Omega)$, define

$$\Phi(r,\theta,z) = \varphi(r,z) \qquad \text{in } \Omega^* \ \{r = 0\}$$

$$\Phi(0,\theta,z) = 0 \quad .$$

Then we have

$$|\langle \sigma^*_{\theta z}, \varphi \rangle_{\Omega,r}| = |\frac{1}{2\pi} \int_{\Omega^*} \sigma^0_{\theta z} \Phi \, dV|$$

$$\leq \frac{1}{2\pi} \|\sigma^0_{\theta z}\|_{L^2(\Omega^*)} \cdot \|\Phi\|_{L^2(\Omega^*)} = \frac{1}{\sqrt{2\pi}} \|\sigma^0_{\theta z}\|_{L^2(\Omega^*)} \cdot \|\varphi\|_{L^2_r(\Omega)} .$$

Hence $\sigma^*_{\theta z}$ (more precisely, its restriction in Ω) belongs to the dual space of $L^2_r(\Omega)$ and then

$$\sigma^*_{\theta z} \in L^2_r(\Omega) .$$

So

$$\int_{\Omega^*} (\sigma^*_{\theta z})^2 dV = \int_0^{2\pi} d\theta \int_\Omega (\sigma^*_{\theta z})^2 r\,dr\,dz = 2\pi \|\sigma^*_{\theta z}\|^2_{L^2_r(\Omega)} < \infty$$

i.e. $\sigma^*_{\theta z} \in L^2(\Omega^*)$. Similarly, we have

$$v \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \sigma^0_{\theta z}}{\partial r} d\theta \in L^2(\Omega^*) .$$

Given $\varphi \in C^\infty_0(\Omega)$, we have

$$\langle v, \varphi \rangle_{\Omega,r} = \int_\Omega \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \sigma^0_{\theta z}}{\partial r} d\theta\right) \varphi\, r\,dr\,dz$$

$$= \frac{1}{2\pi} \int_{\Omega^*} \frac{\partial \sigma^0_{\theta z}}{\partial r} \varphi \, dV = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_\Omega \frac{\partial \sigma^0_{\theta z}}{\partial r} \varphi\, r\,dr\,dz$$

$$= -\frac{1}{2\pi} \int_0^{2\pi} d\theta \int_\Omega \sigma^0_{\theta z} \frac{\partial(\varphi r)}{\partial r} dr\,dz$$

$$= -\frac{1}{2\pi} \int_\Omega \left(\int_0^{2\pi} \sigma^0_{\theta z} d\theta\right) \frac{\partial(\varphi r)}{\partial r} dr\,dz$$

$$= -\int_\Omega \sigma^*_{\theta z} \frac{\partial(\varphi r)}{\partial r} dr\,dz = \int_\Omega \frac{\partial \sigma^*_{\theta z}}{\partial r} \varphi\, r\,dr\,dz$$

$$= \langle \frac{\partial \sigma^*_{\theta z}}{\partial r}, \varphi \rangle_{\Omega,r} .$$

It means that

$$\frac{\partial \sigma^*_{\theta z}}{\partial r} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \sigma^0_{\theta z}}{\partial r} d\theta \in L^2(\Omega^*) .$$

Similarly, we obtain

$$\frac{\partial\sigma^*_{\theta z}}{\partial z} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial\sigma^0_{\theta z}}{\partial z}\, d\theta \in L^2(\Omega^*) \quad .$$

But

$$\frac{\partial\sigma^*_{\theta z}}{\partial\theta} = 0 \quad .$$

Hence $\sigma^*_{\theta z} \in H^1(\Omega^*)$. The same argument indicates that $\sigma^*_{r\theta} \in H^1(\Omega^*)$. and we obtain

$$\sigma^* \in [H^1(\Omega^*)]^6 \quad .$$

Turn to (3.3) - (3.6). Check the second equations of (3.3) and (3.6) for σ^*:

$$\frac{\partial\sigma^*_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma^*_{\theta}}{\partial\theta} + \frac{\partial\sigma^*_{\theta z}}{\partial z} + \frac{2\sigma^*_{r\theta}}{r} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial\sigma^0_{r\theta}}{\partial r} + \frac{\partial\sigma^0_{\theta z}}{\partial z} + \frac{2\sigma^0_{r\theta}}{r}\right)d\theta$$

$$= -\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{r}\frac{\partial\sigma^0_{\theta}}{\partial\theta}\, d\theta \qquad \text{(since (3.3))}$$

$$= 0 \qquad \text{(periodicity of } \sigma \text{ in } \theta)$$

$$\int_{\Gamma^*_{22}} r\sigma^*_{\theta z}\, dS = \int_{\Gamma^*_{22}} r\left(\frac{1}{2\pi}\int_0^{2\pi}\sigma^0_{\theta z}\, d\theta\right) r\,d\theta'\,dr$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta' \int_0^{R(L)} dr \int_0^{2\pi} r^2\sigma^0_{\theta z}\, d\theta$$

$$= \int_0^{R(L)} dr \int_0^{2\pi} r^2\sigma^0_{\theta z}\, d\theta = \int_{\Gamma^*_{22}} r\sigma^0_{\theta z}\, dS = T \quad .$$

The rest of (3.3) - (3.6) is clear.

Now we prove

$$F(\sigma^*) = (\sigma^*_{\theta z})^2 + (\sigma^*_{r\theta})^2 - k^2 \le 0 \quad . \tag{4.4}$$

We have

$$(\sigma^*_{\theta z})^2 + (\sigma^*_{r\theta})^2 = \sigma^*_{\theta z}\sigma^*_{\theta z} + \sigma^*_{r\theta}\sigma^*_{r\theta}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (\sigma^0_{\theta z}\sigma^*_{\theta z} + \sigma^0_{r\theta}\sigma^*_{r\theta})\, d\theta$$

$$\leq \frac{1}{2\pi} \left(\int_0^{2\pi} ((\sigma^*_{\theta z})^2 + (\sigma^*_{r\theta})^2)\, d\theta\right)^{1/2} \cdot \left(\int_0^{2\pi} ((\sigma^0_{\theta z})^2 + (\sigma^0_{r\theta})^2)\, d\theta\right)^{1/2}$$

$$= \frac{1}{\sqrt{2\pi}} [(\sigma^*_{\theta z})^2 + (\sigma^*_{r\theta})^2]^{1/2} \cdot \left(\int_0^{2\pi} [(\sigma^0_{\theta z})^2 + (\sigma^0_{r\theta})^2]\, d\theta\right)^{1/2}$$

i.e.

$$(\sigma^*_{\theta z})^2 + (\sigma^*_{r\theta})^2 \leq \frac{1}{2\pi} \int_0^{2\pi} [(\sigma^0_{\theta z})^2 + (\sigma^0_{r\theta})^2]\, d\theta \quad . \tag{4.5}$$

On the other hand, we have

$$(\sigma^0_{\theta z})^2 + (\sigma^0_{r\theta})^2 \leq F(\sigma^0) + k^2 \leq k^2 \quad .$$

Therefore, we obtain

$$(\sigma^*_{\theta z})^2 + (\sigma^*_{r\theta})^2 \leq \frac{1}{2\pi} \int_0^{2\pi} k^2 d\theta = k^2$$

i.e., (4.4) is valid, and $\sigma^* \in M_2$.

Finally, we compute $J_1(\sigma^0) - J_1(\sigma^*)$. Clearly,

$$(\sigma^*)^T A\sigma^* = \frac{2(1+\nu)}{E} [(\sigma^*_{\theta z})^2 + (\sigma^*_{r\theta})^2]$$

$$(\sigma^0)^T A\sigma^0 = \frac{1}{E} [(\sigma^0_r)^2 + (\sigma^0_\theta)^2 + (\sigma^0_z)^2 - 2\nu(\sigma^0_r\sigma^0_\theta + \sigma^0_r\sigma^0_z + \sigma^0_\theta\sigma^0_z)]$$

$$+ \frac{2(1+\nu)}{E} [(\sigma^0_{r\theta})^2 + (\sigma^0_{\theta z})^2 + (\sigma^0_{rz})^2] \geq \frac{2(1+\nu)}{E} [(\sigma^0_{\theta z})^2 + (\sigma^0_{r\theta})^2]$$

$$(\text{since } 0 < \nu < \tfrac{1}{2}) \quad .$$

It follows from (4.5) that

$$\int_{\Omega^*} [(\sigma^*_{\theta z})^2 + (\sigma^*_{r\theta})^2]\, dV \leq \int_{\Omega^*} [(\sigma^0_{\theta z})^2 + (\sigma^0_{r\theta})^2]\, dV \quad .$$

Hence

$$2(J_1(\sigma^0) - J_1(\sigma^*)) = \int_{\Omega^*} [(\sigma^0)^T A\sigma^0 - (\sigma^*)^T A\sigma^*]\, dV$$

$$\geq \frac{2(1+\nu)}{E} \left(\int_{\Omega^*} [(\sigma^0_{\theta z})^2 + (\sigma^0_{r\theta})^2]\, dV - \int_{\Omega^*}[(\sigma^*_{\theta z})^2 + (\sigma^*_{r\theta})^2]\, dV\right) \geq 0$$

(4.3) has been proved, and the proof of the theorem is completed.

Q.E.D.

The basic idea of the above proof is the same as that of the so-called semi-inverse method.

This theorem enables us to take the set

$$N = \{\sigma \in M_2 : \sigma_r = \sigma_\theta = \sigma_z = \sigma_{rz} = 0 \text{ in } \Omega^*\}$$

as the set of the admissible stress vectors instead of M_2 in problem (A). Then we have the following problem.

Problem (B). Find $\sigma^0 \in N$ such that

$$J_1(\sigma^0) = \min_{\sigma \in N} J_1(\sigma) \quad .$$

Obviously, problem (B) has at most one solution; if σ^0 is the solution of problem (A), then it is also the solution of problem (B).

Remark 4.1. If $\sigma \in N$, then it follows from (3.3) that

$$\frac{\partial \sigma_{\theta z}}{\partial \theta} = \frac{\partial \sigma_{r\theta}}{\partial \theta} = 0 \quad .$$

Therefore,

$$J_1(\sigma) = \frac{(1+\nu)2\pi}{E} J_0(\sigma)$$

where

$$J_0(\sigma) = \int_\Omega (\sigma_{r\theta}^2 + \sigma_{\theta z}^2) r \, dr dz \quad .$$

5. The variational problem for stress function

If problem (B) has solution $\sigma^0 \in [C^0(\overline{\Omega^*})]^6$, then we have by remark 4.1:

$$\sigma^0 \in N_1, \ J_0(\sigma^0) = \min_{\sigma \in N_1} J_0(\sigma) \qquad \text{(C)}$$

where N_1 is a subset of N:

$$N_1 = \{\sigma \in N : \sigma \in [C^0(\overline{\Omega^*})]^6\}$$

$\forall \sigma \in N_1$, by virtue of (3.3) we have

$$\frac{\partial}{\partial r}(r^2 \sigma_{r\theta}) + \frac{\partial}{\partial z}(r^2 \sigma_{\theta z}) = 0 \quad .$$

Then it is easy to show that there exists $v^* \in H^2(\Omega) \cap C^1(\overline{\Omega})$ such that

$$\frac{\partial v^*}{\partial r} = r^2\sigma_{\theta z}, \frac{\partial v^*}{\partial z} = -r^2\sigma_{r\theta} \quad \text{in } \bar{\Omega} \ .$$

Hence $\frac{\partial v^*}{\partial z} = 0$ on Γ_0, and $v^* = C_1$ on Γ_0. Let $v = v^* - C_1$. Then

$$\frac{\partial v}{\partial r} = r^2\sigma_{\theta z}, \frac{\partial v}{\partial z} = -r^2\sigma_{r\theta} \quad \text{in } \bar{\Omega} \tag{5.1}$$

$$v = 0 \qquad\qquad \text{on } \Gamma_0 \tag{5.2}$$

$$J_0(\sigma) = \int_\Omega r^{-3}[(\frac{\partial v}{\partial r})^2 + (\frac{\partial v}{\partial z})^2]drdz \equiv J(v) \ . \tag{5.3}$$

It follows from (3.4) and (5.1) that

$$\frac{dv}{ds} = -\frac{\partial v}{\partial z}\cos(n,r) + \frac{\partial v}{\partial r}\cos(n,z) = 0 \qquad \text{on } \Gamma_1$$

where s is the tangential direction of Γ_1. Therefore,

$$v = C_2 \qquad \text{on } \Gamma_1 \ . \tag{5.4}$$

Since (3.6), (5.1) and (5.2), we have

$$T = \int_0^{2\pi} d\theta \int_0^{R(0)} r^2\sigma_{\theta z}drd\theta = 2\pi \int_0^{R(0)} \frac{\partial v}{\partial r}dr = 2\pi\, v(0,R(0)) \ , \tag{5.5}$$

Combine (5.4) and (5.5) we obtain

$$v = T/2\pi \qquad \text{on } \Gamma_1 \ . \tag{5.6}$$

By (3.5) and (5.1) we have

$$\frac{\partial v}{\partial n} = 0 \qquad \text{on } \Gamma_2 \ . \tag{5.7}$$

It follows from $F(\sigma) \le 0$ that

$$|\nabla v|^2 = r^4(\sigma_{r\theta}^2 + \sigma_{\theta z}^2) \le k^2r^4$$

$$|\nabla v| \le kr^2 \qquad \text{in } \Omega \ . \tag{5.8}$$

Since $\sigma \in [H^1(\Omega^*)]^6$, we have

$$\sigma_{r\theta}, \sigma_{\theta z} \in H_r^1(\Omega)$$

where

$$H_r^1(\Omega) = \{v \in L_r^2(\Omega) : \frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \in L_r^2(\Omega)\}$$

$L_r^2(\Omega)$ is defined by (4.3).

Hence

$$r^{-2}\frac{\partial v}{\partial r}, r^{-2}\frac{\partial v}{\partial z} \in H_r^1(\Omega) \ , \tag{5.9}$$

particularly,

$$\int_\Omega r^{-3}\left(\frac{\partial v}{\partial r}\right)^2 drdz,\ \int_\Omega r^{-3}\left(\frac{\partial v}{\partial z}\right)^2 drdz < \infty \quad . \tag{5.10}$$

Finally by (5.1) and (5.2) we have

$$v^2 = \left(\int_0^r r^2\sigma_{\theta z}dr\right)^2 \leq \int_0^r r^4 dr \cdot \int_0^r \sigma_{\theta z}^2 dr \leq Cr^5$$

so

$$\int_\Omega r^{-5}v^2 drdz < \infty \quad . \tag{5.11}$$

Now let

$$N_2 = \{v : v \in H^2(\Omega) \cap C^1(\overline{\Omega}),\ \ (5.2) \text{ and } (5.6) - (5.9) \text{ are valid}\} \quad .$$

Then it is easy to see that (5.1) - (5.2) defines a biunivocal map from N_1 onto N_2, and problem (C) is equivalent to the following problem:

$$v_0 \in N_2,\quad J(v_0) = \min_{v \in N_2} J(v) \tag{D}$$

where (noting (5.3))

$$J(v) = \int_\Omega \rho\left[\left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2\right] drdz \tag{5.12}$$

$$\rho = r^{-3} \quad . \tag{5.13}$$

Now we enlarge the set of admissible functions of the variational problem (D) for solving the problem on the existence of the solution. Noting (5.11) and that only the derivatives of first order appear in the functional $J(v)$, we introduce a set as follows

$$K = \{v : v \in H^1_\rho(\Omega),\ v = T/2\pi \text{ on } \Gamma_1,\ |\nabla v| \leq kr^2 \text{ in } \Omega\} \tag{5.14}$$

where

$$H^1_\rho(\Omega) = \{v \in L^2_\rho(\Omega) : \frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \in L^2_\rho(\Omega)\} \tag{5.15}$$

$$L^2_\rho(\Omega) = \{v : v \text{ measurable},\ \|v\|_{L^2_\rho(\Omega)} < +\infty\} \tag{5.16}$$

with norm, respectively,

$$\|v\|_{L^2_\rho(\Omega)} = \left[\int_\Omega \rho v^2 drdz\right]^{1/2}$$

$$\|v\|_{H^1_\rho(\Omega)} = \left(\|v\|^2_{L^2_\rho(\Omega)} + \left\|\frac{\partial v}{\partial r}\right\|^2_{L^2_\rho(\Omega)} + \left\|\frac{\partial v}{\partial z}\right\|^2_{L^2_\rho(\Omega)}\right)^{1/2} ,$$

and $v = T/2\pi$ is in the sense of trace (see, for instance, Nečas [1967, p. 15]). K is a natural extension of N_2 because of that according to theorem 2.2 in Cryer [1980], if $u \in H^1_\rho(\Omega)$, then (5.2) (in the sense of trace) and (5.11) are valid. Therefore, we have the variational problem for stress function as follows.

Problem (E). Find $v_0 \in K$ such that

$$J(v_0) = \min_{v \in K} J(v)$$

and, equivalently,

Problem (F). Find v_0 such that

$$\begin{cases} v_0 \in K \\ a(v_0, v-v_0) \geq 0, \quad \forall\, v \in K \end{cases}$$

where

$$a(v', v") = \int_\Omega \rho\left(\frac{\partial v'}{\partial r}\frac{\partial v"}{\partial r} + \frac{\partial v'}{\partial z}\frac{\partial v"}{\partial z}\right) dr dz \quad .$$

Similarly to the case of constant cross-section, we introduce the obstacle problems relevant to problem (F). There are two obstacle problems to be considered.

Problem (F1). Find v_1 such that

$$\begin{cases} v_1 \in K_1 \\ a(v_1, v-v_1) \geq 0 \quad \forall\, v \in K_1 \end{cases}$$

where

$$K_1 = \{v \in H^1_\rho(\Omega) : v = T/2\pi \text{ on } \Gamma_1, v \geq \psi_1 \text{ in } \Omega\} \quad ,$$

and ψ_1 is the solution of the Cauchy problem

$$\psi_1 \in C^2(\bar\Omega)$$

$$|\nabla\psi_1|^2 = k^2r^4,\ \psi_1 < T/2\pi \quad \text{in} \quad \Omega \tag{5.17}$$

$$\psi_1 = T/2\pi \qquad\qquad \text{on} \quad \Gamma_1 \quad .$$

<u>Problem (F2)</u>. Find v_2 such that

$$\begin{cases} v_2 \in K_2 \\ a(v_2,\ v-v_2) \geq 0 \qquad \forall\ v \in K_2 \end{cases}$$

where

$$K_2 = \{v \in H^1_\rho(\Omega) \ :\ v = T/2\pi \ \text{ on } \ \Gamma_1,\ v \leq \psi_2 \ \text{ in } \ \Omega\} \quad ,$$

and ψ_2 is the solution of the Cauchy problem

$$\psi_2 \in C^2(\bar\Omega)$$

$$|\nabla\psi_2|^2 = k^2r^4,\ \psi_2 > 0 \quad \text{in} \quad \Omega \tag{5.18}$$

$$\psi_2 = 0 \qquad\qquad \text{on} \quad \Gamma_0 \quad .$$

The problem (F1) is just the problem (4.7) in Cryer [1980], there the solution for (5.17) is also discussed.

6. ψ_2 <u>- solution of the Cauchy problem (5.18)</u>

Assume ψ_2 is the solution of the Cauchy problem (5.18). Let $p = \frac{\partial\psi_2}{\partial z}$, $q = \frac{\partial\psi_2}{\partial r}$. Then the equation is

$$F \equiv p^2 + q^2 - k^2r^4 = 0 \quad .$$

We have along the characteristics parametrized by s in $\bar\Omega\backslash\Gamma_0$ (Courant and Hilbert [1962, p. 78])

$$\frac{dz}{ds} = \frac{\partial F}{\partial p} = 2p \tag{6.1}$$

$$\frac{dr}{ds} = \frac{\partial F}{\partial q} = 2q \tag{6.2}$$

$$\frac{d\psi_2}{ds} = p\,\frac{\partial F}{\partial p} + q\,\frac{\partial F}{\partial q} = 2(p^2 + q^2) = 2k^2r^4 \tag{6.3}$$

$$\frac{dp}{ds} = -(p\,\frac{\partial F}{\partial \psi_2} + \frac{\partial F}{\partial z}) = 0 \tag{6.4}$$

$$\frac{dq}{ds} = -(q\,\frac{\partial F}{\partial \psi_2} + \frac{\partial F}{\partial r}) = 4k^2r^3 \quad . \tag{6.5}$$

Since $\psi_2 \in C^2(\bar{\Omega})$, (6.1) - (6.5) are also valid on Γ_0. Given $(z_0,0) \in \Gamma_0$, consider the characteristic passing this point. Let the parameter value of this point be $s = 0$. We have initial conditions (since $\psi_2 = 0$ on Γ_0 and $F = 0$)

$$z(0) = z_0,\ r(0) = p(0) = q(0) = \psi_2(0) = 0 \quad .$$

It follows from (6.4) that $p(s) \equiv 0$. Hence $z \equiv z_0$ (since (6.1)), and $q = kr^2$ (since $F = 0$ and $\psi_2 \geq 0$). Therefore, we obtain by integrating along the characteristic

$$\psi_2(z_0,r) = \psi_2(s) = \int_0^s \frac{d\psi_2}{ds}\,ds = \int_0^s \left[\frac{\partial \psi_2}{\partial z}\,\frac{dz}{ds} + \frac{\partial \psi_2}{\partial r}\,\frac{dr}{ds}\right] ds$$

$$= \int_0^s q\,\frac{dr}{ds}\,ds = \int_0^r q\,dr = \int_0^r kr^2 dr = kr^3/3 \quad .$$

Since $(z_0,0) \in \Gamma_0$ is arbitrary, we obtain the solution

$$\psi_2(z,r) = kr^3/3 \quad \text{in } \bar{\Omega} \quad . \tag{6.6}$$

7. Properties of the set K

Denote by $C^{0,1}(\bar{\Omega})$ the set of functions which satisfy Lipschitz conditions, that is, if $v \in C^{0,1}(\bar{\Omega})$, then

$$h_1(v,\bar{\Omega}) \equiv \sup_{\substack{p_1,p_2 \in \bar{\Omega} \\ p_1 \neq p_2}} \frac{|v(p_1) - v(p_2)|}{|p_1 - p_2|} < +\infty \tag{7.1}$$

where $p_1 = (z_1, r_1)$, $p_2 = (z_2, r_2)$, and $|p_1 - p_2| = [(z_1 - z_2)^2 + (r_1 - r_2)^2]^{1/2}$.

<u>Lemma 7.1</u>. The following statements are equivalent:

(a) $v \in C^{0,1}(\bar{\Omega})$

(b) $v \in H^1(\Omega)$, and $\frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \in L^\infty(\Omega)$.

<u>Proof</u>: It has been shown (cf. Adams [1975, pp. [illegible]) that if v, $\frac{\partial v}{\partial r}$, $\frac{\partial v}{\partial z} \in L^\infty(\Omega)$ then

$$h_1(v,\Omega) \le C_1(\|v\|_{L^\infty(\Omega)} + \|\frac{\partial v}{\partial r}\|_{L^\infty(\Omega)} + \|\frac{\partial v}{\partial z}\|_{L^\infty(\Omega)}) \quad . \tag{7.2}$$

By similar argument we can prove that if $\frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \in L^\infty(\Omega)$, and $v \in L^1(\Omega)$ then

$$h_1(v,\Omega) \le C_2 \left(\|\frac{\partial v}{\partial r}\|_{L^\infty(\Omega)} + \|\frac{\partial v}{\partial z}\|_{L^\infty(\Omega)}\right) \quad . \tag{7.3}$$

Therefore, it is clear that (b) implies (a).

Now assume $v \in C^{0,1}(\bar{\Omega})$. Then v possesses a total differential a.e. in Ω (Morrey [1966, p. 65]), i.e.

$$v(p_1) - v(p_2) = [\frac{\partial v}{\partial z}](z_1 - z_2) + [\frac{\partial v}{\partial r}](r_1 - r_2) + 0\,(|p_1 - p_2|) \tag{7.4}$$

$$\text{a.e. in } \Omega$$

where $[\frac{\partial v}{\partial z}]$, $[\frac{\partial v}{\partial r}]$ are the partial derivatives in the usual sense. Clearly, they are measurable in Ω. It follows from (7.1) and (7.4) that

$$|[\frac{\partial v}{\partial z}]|,\ |[\frac{\partial v}{\partial r}]| \le h_1(v,\bar{\Omega}) \quad \text{a.e. in } \Omega \quad . \tag{7.5}$$

Hence $v \in H^1(\Omega)$ and (Morrey [1966, p. 63])

$$[\frac{\partial v}{\partial z}] = \frac{\partial v}{\partial z},\ [\frac{\partial v}{\partial r}] = \frac{\partial v}{\partial r} \quad . \tag{7.6}$$

Then by (7.5) we obtain $\frac{\partial v}{\partial z}, \frac{\partial v}{\partial r} \in L^\infty(\Omega)$, i.e. (b) is valid.

Q.E.D.

<u>Proposition 7.2</u>. $K \subset K_1 \cap K_2$.

Proof: Given $v \in K$, it is enough to prove that

$$\psi_1 \leq v \leq \psi_2 \quad . \tag{7.7}$$

By Lemma 7.1 we have $v \in C^{0,1}(\bar{\Omega})$. Then (7.6) is valid. Since $v \in H^1(\Omega)$, is absolutely continuous in r on $0 \leq r \leq R(z)$ for almost all value z (Morrey [1966, p. 66]). Therefore, noting that $v = 0$ on Γ_0, we obtain

$$v(r,z) \leq |v(r,z)| = |\int_0^r [\frac{\partial v}{\partial r}] dr| = |\int_0^r \frac{\partial v}{\partial r} dr|$$

$$\leq \int_0^r |\nabla v| dr \leq \int_0^r kr^2 dr = \psi_2 \quad \text{a.e. in } \Omega \quad .$$

The second part of (7.7) has been proved. Now prove the first part. The system of the characteristic equations for the Cauchy problem (5.17) has the same form as (6.1) - (6.5). Let the parameter value $s = 0$ correspond to the point on Γ_1. Then it follows from (6.2) that the point in Ω corresponds to the negative value of s. Noting that v has a total differential a.e. in Ω and that (7.6) is valid, we have along the characteristic

$$v(z(0),r(0)) - v(z(s),r(s)) = \int_s^0 \frac{dv}{ds} ds = \int_s^0 (\frac{\partial v}{\partial z} \frac{dz}{ds} + \frac{\partial v}{\partial r} \frac{dr}{ds}) ds$$

$$= \int_s^0 (\frac{\partial v}{\partial z} 2p + \frac{\partial v}{\partial r} 2q) ds \leq 2 \int_s^0 |\nabla v| \cdot (p^2 + q^2)^{1/2} ds$$

$$\leq 2 \int_s^0 k^2 r^4 ds = \int_s^0 \frac{d\psi_1}{ds} ds = \psi_1(z(0),r(0)) - \psi_1(z(s),r(s)) \quad ,$$

But $v(z(0),r(0)) = \psi_1(z(0),r(0)) = T/2\pi$. Hence $v \geq \psi_1$.

Q.E.D.

Proposition 7.3. The sets K, K_1 and K_2 are closed, convex subsets in $H^1_\rho(\Omega)$.

Proof: Given $v, w \in K$ and λ with $0 \leq \lambda \leq 1$, we have

$$|\nabla[\lambda v + (1-\lambda)w]| = [(\lambda \frac{\partial v}{\partial z} + (1-\lambda) \frac{\partial w}{\partial z})^2 + (\lambda \frac{\partial v}{\partial r} + (1-\lambda) \frac{\partial w}{\partial r})^2]^{1/2}$$

$$\leq [(\lambda \frac{\partial v}{\partial z})^2 + (\lambda \frac{\partial v}{\partial r})^2]^{1/2} + [(1-\lambda)^2(\frac{\partial w}{\partial z})^2 + (1-\lambda)^2(\frac{\partial w}{\partial r})^2]^{1/2}$$

$$= \lambda|\nabla v| + (1-\lambda)|\nabla w| \leq kr^2 \quad .$$

By the linearity of the trace operator (Nečas [1967, p. 15]) we have

$$\lambda v + (1-\lambda)w = \lambda T/2\pi + (1-\lambda)T/2\pi = T/2\pi \quad \text{on} \quad \Gamma_1 \quad .$$

Hence $\lambda v + (1-\lambda)w \in K$ and K is convex.

Let $\{v_n\}$ be a Cauchy sequence in K. Since $H_\rho^1(\Omega)$ is a Banach space (Cryer [1980]), there exists $v \in H_\rho^1(\Omega)$ such that $v_n \to v$ in $H_\rho^1(\Omega)$. It follows from the continuity of trace operator (Nečas [1967, p. 15]) that $v = T/2\pi$ on Γ_1. By well-known subsequence argument we obtain that $|\nabla v| \leq kr^2$ in Ω. Then K is closed.

The conclusion about K_1 and K_2 can be proved by similar argument.

Q.E.D.

8. Solution of the problem (F)

At first we solve the problem (F_2) it suggests the solution of the problem (F). Let (cf. Cryer [1980, Remark 5.3])

$$\overline{R} = \min_{0 \leq z \leq L} R(z) \tag{8.1}$$

$$k_0 = (3T/2\pi)\overline{R}^{-3} \quad . \tag{8.2}$$

We need a lemma it may easily be shown by a well-know theorem (Adams [1975, p. 54]).

Lemma 8.1. If $u \in C^0(\overline{\Omega}) \cap H^1(\Omega)$, then

$$\text{tr } u = u \qquad \text{on} \quad \partial\Omega$$

where tr u is the trace of u on $\partial\Omega$.

Proposition 8.2. If $k < k_0$ then the problem (F_2) has no solution. If $k \geq k_0$ then it has a unique solution.

Proof: There exists a $\bar{z}$ such that

$$0 < \bar{z} < L, \qquad R(\bar{z}) = \bar{R} \quad .$$

If $k < k_0$ then by (6.6) and (8.2) we have

$$\psi_2(\bar{z},\bar{R}) = k\bar{R}^3/3 < k_0\bar{R}^3/3 = T/2\pi \quad . \tag{8.3}$$

Thus, there exists a real number c and an open neighborhood Σ of the point $(\bar{z},\bar{R})$ such that

$$\psi_2 < c < T/2\pi \text{ in } \Sigma_1 \equiv \Sigma \cap \Omega \quad .$$

Assume that K_2 is nonempty. Take $v \in K_2$. Then

$$v < c < T/2\pi \text{ in } \Sigma_1 \quad ,$$

and there exists a sequence $\{v_n\} \subset C^\infty(\bar{\Omega})$ such that

$$\|v_n - v\|_{H^1(\Omega)} \to 0 \quad (n \to \infty) \quad . \tag{8.4}$$

From the construction of v_n (Adams [1975, pp. 54-56]) we see that there exists $\Sigma_2 \subset \Sigma_1$ and a real number $c^* > c$ such that meas $\Gamma^* > 0$, where $\Gamma^* = \Gamma_1 \cap \bar{\Sigma}_2$, and that

$$v_n < c^* < T/2\pi \text{ in } \bar{\Sigma}_2 \quad . \tag{8.5}$$

Then by the continuity of trace operator and (8.4) we have

$$\|v_n - \text{tr } v\|_{L_2(\Gamma^*)} < \|v_n - \text{tr } v\|_{L_2(\Gamma_1)} < C_1\|v_n - v\|_{H^1(\Omega)} \to 0 \quad .$$

Hence there exists a subsequence $\{v_n'\}$ which converge to $\text{tr } v$ a.e. on Γ^*, and by (8.5) we obtain

$$\text{tr } v < c^* < T/2\pi \text{ on } \Gamma^* \quad .$$

This contradicts that $\text{tr } v = T/2\pi$ on Γ_1. Therefore, K_2 is empty, and the problem (F_2) has no solution.

If $k > k_0$, then let

$$v = \min(\psi_2, T/2\pi) \text{ in } \bar{\Omega} \quad . \tag{8.6}$$

Show that $v \in K_2$. Clearly, $v < \psi_2$, and

$$v = kr^3/3 \qquad \text{in } \Omega \cap \{r < d\}$$

$$v = T/2\pi \qquad \text{in } \Omega \cap \{r > d\}$$

where $d = (3T/2\pi\ k)^{1/3}$. So we have (Gilbarg and Trudinger [1977, p. 145])

$$
\begin{aligned}
v \in H^1(\Omega), \quad \frac{\partial v}{\partial z} &= 0 \qquad \text{in } \Omega \\
\frac{\partial v}{\partial r} &= kr^2 \qquad \text{in } \Omega \cap \{r < d\} \qquad\qquad (8.7)\\
\frac{\partial v}{\partial r} &= 0 \qquad \text{in } \Omega \cap \{r > d\} \quad .
\end{aligned}
$$

It is easy to see that

$$\|v\|_{H^1_\rho(\Omega)} < \infty \quad .$$

Hence $v \in H^1_\rho(\Omega)$. Clearly, $v \in C^0(\bar{\Omega}) \cap H^1(\Omega)$, and $v = T/2\pi$ on Γ_1 in the usual sense. So $v = T/2\pi$ on Γ_1 in the sense of trace of lemma 8.1, and $v \in K_2$.

Thus, K_2 is a closed, convex, nonempty set of $H^1_\rho(\Omega)$; and $a(v',v'')$ is a continuous, coercive, real bilinear form on $H^1_\rho(\Omega) \times H^1_\rho(\Omega)$ (Cryer [1980, p. 549]); hence the problem (F_2) has unique solution (Stampacchia [1964]).

Q.E.D.

<u>Theorem 8.3</u>. If $k < k_0$ then the problem (F) has no solution. If $k > k_0$ then it has a unique solution.

<u>Proof</u>: If $k < k_0$ then K_2 is empty. By proposition 7.2 K is also empty, and the problem (F) has no solution.

If $k > k_0$, then take v as in (8.6). We have known that $v \in H^1_\rho(\Omega)$ and $v = T/2\pi$ on Γ_1. But from (8.7) we see that $|\nabla v| \le kr^2$ in Ω. Hence $v \in K$, and K is nonempty. By the similar argument to that in the proof of Proposition 8.2 we obtain that the problem (F) has a unique solution.

Q.E.D.

<u>Remark 8.1</u>. By virtue of theorem 8.3 and Proposition 7.2 we obtain that the theorem 6.2 in Cryer [1980] means that the problem (F) and (F_1) are equivalent under the conditions described there.

Remark 8.2. The conjecture that the problems (F) and (F_2) are equivalent is not right. The numerical experiment we have made for the case $R(z) \equiv 1$ indicates that $v_0 \neq v_1$. This fact may be shown by analytical method in this case.

REFERENCES

Adams, R. A.: Sobolev Spaces, Academic Press, New York (1975).

Brezis, H. and Sibony, M.: Equivalence de deux inéquations variationnelles, Arch. Rat. Mech. Anal. 41, 254-265 (1971).

Courant, R. and Hilbert, D.: Methods of Mathematical Physics, v. 2, Interscience, New York (1962).

Cryer, C. W.: The solution of the axisymmetric elastic-plastic torsion of a shaft using variational inequalities, J. Math. Anal. Appl. 76, 535-570 (1980).

Eddy, R. P. and Shaw, F. S.: Numerical solution of elastoplastic torsion of a shaft of rotational symmetry, J. Appl. Mech. 16, 139-148 (1949).

Friedman, A.: The unloading problem in elasto-plasticity, Proc. Sem. Pavia (1979), v. 2, 277-280, Tecnoprint s.n.c., Roma (1980).

Gilbarg, D. and Trudinger, N. S.: Elliptic partial differential equations of second order, Springer-Verlag, New York (1977).

Lanchon, H.: Torsion élastoplastique d'un arbre cylindrique de section simplement on multiplement connexe, J. Meca. 13, 267-320 (1974).

Martin, J. B.: Plasticity: fundamentals and general results, The MIT Press, Cambridge, Massachusetts (1975).

Morrey, C. B. Jr.,: Multiple integrals in the calculus of variations, Springer-Verlag, New York (1966).

Nečas, J.: Les méthodes directes en théorie de équation elliptiques, Academia, Prague (1967).

Pozzi, G. A.: Some recent results for the elastic-plastic torsion problem, Proc. Sem. Pavia (1979), v. 2, 479-485, Tecnoprint s.n.c., Roma (1980).

Stampacchia, G.: Formes bilineaires coercitives sur les ensembles convexes, C. R. Acad. Sci. Paris, 258, 4413-4416 (1964).

Timoshenko, S. and Goodier, J. N.: Theory of elasticity, second edition, McGraw-Hill Book Company, Inc., New York (1951).

Ting, T. W.: Elastic plastic torsion of simply connected cylindrical bars, Indiana Univ. Math. J. 10, 1047-1076 (1971).

Ting. T. W.: Elastic-plastic torsion of cylindrical pipes, Lecture Notes in Math. 503, 521-531, Springer-Verlag, New York (1976).

Varga, R.: Matrix iterative analysis, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1962).

SZZ/jvs

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2270	2. GOVT ACCESSION NO. AD-A110479	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On An Axisymmetric Free Boundary Problem		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Zhu-Zi Zhou		8. CONTRACT OR GRANT NUMBER(s) MCS77-26732 DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS (see Item 18 below)		12. REPORT DATE August 1981
		13. NUMBER OF PAGES 25
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office and National Science Foundation
P. O. Box 12211 Washington, DC 20550
Research Triangle Park
North Carolina 27709

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

torsion; elastic-plastic; axisymmetric; free boundary problem; variational inequalities; Haar-Kármán principle.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The axisymmetric elastic-plastic torsion of a shaft of general shape subject to the Hencky consistency condition with the von Mises yield function is considered. It is proved that the Haar-Kármán principle is valid in this case, and that the problem is essentially two-dimensional. The problem is reformulated as a variational inequality, and the existence and uniqueness of the solution is studied.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

END

DATE

FILMED

3-82

DTIC